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RESTRICTIONS ON NON-ASSOCIATED PLASTIC FLOW LAWS IMPOSED BY THERMODYNAMICS AND UNIQUENESS

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The restrictions imposed by thermodynamics and uniqueness on non-associated elasto-plastic solids are explored. The postulates of Drucker and Ilyushin are examined and it is shown that they can be violated while the second law of thermodynamics is satisfied. Some of the basic features of non-associated elasto-plastic solids are investigated, including continuity, path dependence and a rather peculiar feature in which the plastic tangent stiffness can exceed the elastic tangent stiffness. The question of uniqueness of initial value problems for non-associated elasto-plastic solids is examined; it is shown that the usual approach for proving uniqueness of associated elasto-plastic solids remains valid even when the flow rule is non-associated. Finally, two types of non-associated plastic models that are widely used by the defense community in ground motion simulation studies are considered in detail, and the implications of the restrictions imposed on these models by thermodynamics are discussed.

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PREFACE

This Topical Report was prepared by S-CUBED for the Defense Nuclear Agency as part of Contract DNA 001-85-C-0024. The research was performed during the period from July 1, 1985 to May 15, 1986. Dr. Murakami, an S-CUBED consultant, is also Associate Professor of Applied Mechanics at the University of California, San Diego.

CONVERSION TABLE

CONVERSION FACTORS FOR U.S. CUSTOMARY TO METRIC (SI) UNITS OF MEASUREMENT

To Convert From	To	Multiply By
angstrom	Meters (m)	1.000 000 x E-10
atmosphere (normal)	Kilo pascal (kPa)	1.013 25 x E +2
bar	Kilo pascal (kPa)	1.000 000 x E +2
barn	meter^2 (m^2)	1.000 000 x E -28
British thermal unit (thermochemical)	joule (J)	1.054 350 x E +3
cal (thermochemical)/cm^2	mega joule/m^2 (MJ/m^2)	4.184 000 x E -2
calorie (thermochemical)	joule (J)	4.184 000
calorie (thermochemical/g	joule per kilogram (J/kg)	4.184 000 x E +3
curies	giga becquerel (Gbq)*	3.700 000 x E +1
degree Celsius	degree kelvin (K)	$t_K = t_K^\circ + 273.15$
degree (angle)	radian (rad)	1.745 329 x E -2
degree Fahrenheit	degree kelvin (K)	$t_K = t_K^\circ + 459.67/1.8$
electron volt	joule (J)	1.602 19 x E -19
erg	joule (J)	1.000 000 x E -7
erg/second	watt (W)	1.000 000 x E -7
foot	meter (m)	3.048 000 x E -1
foot-pound-force	joule (J)	1.355 818
gallon (U.S. liquid)	meter^3 (m^3)	3.785 412 x E -3

* The becquerel (Bq) is the SI unit of radioactivity; 1 Bq = 1 event/s.

CONVERSION TABLE (Continued)

CONVERSION FACTORS FOR U.S. CUSTOMARY TO METRIC (SI) UNITS OF MEASUREMENT (Continued)

To Convert From	To	Multiply By
inch	meter (m)	2.540 000 x E -2
jerk	joule (J)	1.000 000 x E +9
joule kilogram (J/kg) (radiation dose absorbed)	gray (Gy)	1.000 000
kilotons	terajoules	4.183
kip (1000 lbf)	newton (N)	4.448 222 x E +3
kip/inch^2 (ksi)	kilo pascal (kPa)	6.894 757 x E +3
ktap	newton-second/m^2 (N-s/m^2)	1.000 000 x E +2
micron	meter (m)	1.000 000 x E -6
mil	meter (m)	2.540 000 x E -5
mile (international)	meter (m)	1.609 344 x E +3
ounce	kilogram (kg)	2.834 952 x E -2
Pound-force (lbf avoirdupois)	newton (N)	4.448 222
pound-force inch	newton-meter (N•m)	1.129 848 x E -1
pound-force/inch	newton/meter (N/m)	1.751 268 x E +2
pound-force/foot^2	kilo pascal (kPa)	4.788 026 x E -2
pound-force/inch^2 (psi)	kilo pascal (kPa)	6.894 757
pound-mass (lbm avoirdupois)	kilogram kg)	4.535 924 x E -1
pound-mass-foot^2 (moment of inertia)	kilogram-meter^2 (kg•m^2)	4.214 011 x E -2

CONVERSION TABLE (Concluded)

CONVERSION FACTORS FOR U.S. CUSTOMARY TO METRIC (SI) UNITS OF MEASUREMENT (Concluded)

To Convert From	To	Multiply By
pound/mass/foot^3	kilogram-meter^3 (kg/m^3)	1.061 846 x E +1
rad (radiation dose absorbed)	gray (Gy)*	1.000 000 x E -2
roentgen	coulomb/kilogram (C/kg)	2.579 760 x E -4
shake	second (s)	1.000 000 x E -8
slug	kilogram (kg)	1.459 390 x E +1
torr (mm Hg, 0°C)	kilo pascal (kPa)	1.333 22 x E -1

* The gray (Gy) is the accepted SI unit equivalent to the energy imparted by ionizing radiation to a mass and corresponds to one joule/kilogram.

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SECTION 1

INTRODUCTION

Elasto-plastic models with non-associated flow rules are commonly used today in the defense community to describe the responses of soils, and other complex geologic media, to explosive sources. Originally, non-associated flow laws were introduced into such models for the purpose of bringing the predicted volume changes more in line with observed changes. However, in the past several years, it has become increasingly apparent that there are more compelling reasons for using non-associated flow laws in conjunction with soils. First, there is considerable experimental data now which show that plastic flow of granular materials near failure is not associated (Drescher and de Josselin de Jong, 1982; Oda and Konishi, 1974; Drescher, 1976; Lade, Nelson, and Ito, 1987) and, secondly, it has been theoretically established that frictional materials, in general, exhibit non-associated flow laws (Mandel, 1947; Spencer, 1964; Rice, 1976). Inasmuch as soils are frictional materials, in which deformation occurs through the frictional sliding of granular particles, there appears to be both a theoretical and physical basis for adopting non-associated flow laws in modeling soil behavior.

In granular materials the phenomena which invalidate the adoption of associated flow rules are termed "non-normality" and "non-coaxiality". Non-normality implies that plastic strain rate is not normal to the current yield surface. Non-coaxiality means that the principal axes of stress and plastic strain rate do not coincide, which usually leads to the non-normality. An extensive review of microscopic and macroscopic plasticity theories for geomaterials, which exhibit non-normality and non-coaxiality was compiled by Nemat-Nasser (1983). In phenomenological plasticity theories the noncoaxiality has been introduced either by the yield-vertex model (Rudnicki and Rice, 1975) or by a smooth yield surface model with kinematic hardening. In this report, only plastic strain rates expressible by the flow potential are considered together with isotropic hardening of smooth yield surfaces. Therefore, the component of plastic strain rate which represents the effect of non-coaxiality without flow potential is not treated.

Recently, non-associated flow laws have been the subject of considerable discussion and debate in the ground motion community. Specifically, non-associated flow laws have been attacked on the basis that they violate Il'iushin's Postulate and

lead to non-uniqueness of initial value problems. Since Il'iushin's Postulate is not grounded in the laws of thermodynamics, as we shall show in the sequel, its violation is clearly not a critical issue. The question of uniqueness, however, is of concern.

The purpose of this report is to explore the general restrictions that are placed on non-associated flow laws by the second law of thermodynamics and by the requirement that unique solutions exist to initial value problems. The basic question that we seek to address is as follows: Are there any restrictions imposed by either thermodynamics or uniqueness which make non-associated flow laws generally unacceptable for use in conjunction with explosive ground motion studies?

In Section 2, the basic relations from the Onsagerist approach to the thermodynamics of irreversible media are developed. The postulates of Drucker and Il'iushin are discussed in Section 3, where it is shown that they are unrelated to the second law. In Section 4, the basic features of non-associated elasto-plastic models are explored, including continuity, path dependence and a peculiar feature in which the plastic tangent modulus can exceed the corresponding elastic modulus. The question of uniqueness of initial value problems for non-associated elasto-plastic solids is examined in Section 5, where it is shown that the usual approach for proving uniqueness is unsatisfactory when the plastic flow law is non-associated. Finally, in Section 6, two classes of non-associated elasto-plastic models that are presently being used by the defense community are considered, and the implications of the general restrictions imposed by thermodynamics are discussed.

SECTION 2

THERMODYNAMICS OF IRREVERSIBLE MEDIA

In this section, the restrictions imposed by thermodynamics on elasto-plastic constitutive theories having non-associated flow laws are explored. The two main approaches to the thermodynamics of irreversible media are briefly reviewed and difficulties are noted. The approach currently accepted by the majority of researchers in the field, namely, the Onsagerist approach, is adopted in the present study and forms the basis for the developments which follow.

2.1 APPROACHES.

There are essentially two major schools of thought that have emerged in recent years regarding the thermodynamics of continuous media, namely, the Onsagerist irreversible thermodynamics of internal variables (Onsager, 1931; Eckart, 1948; Meixner, 1953; Biot, 1954; Zeigler, 1958; Schapery, 1964; Valanis, 1968, 1971c; Rice, 1971; Nemat-Nasser, 1975a,b) and the rational thermodynamics (Truesdell and Toupin, 1960; Coleman, 1964; Truesdell and Noll, 1965). The essential differences between these two approaches, which have been discussed by Kestin and Rice (1970), Valanis (1971c) and Nemat-Nasser (1975b), lie in the concepts of entropy and temperature. As a result, they lead to different inequalities implied by the second law of thermodynamics (Valanis, 1971c).

In the rational thermodynamics approach, temperature and entropy are assumed to exist strictly as mathematical entities, without physical interpretation, and a Clausius-Duhem inequality is postulated by introducing suitable entropy flux relations. The Onsagerist approach, on the other hand, is basically an extension of classical thermodynamics in which temperature and entropy are considered as physical entities. On this basis, second law considerations lead to a Clausius-Duhem inequality which is different from that postulated by the rational thermodynamicists.

Recently, criticisms have been raised against the rational thermodynamic approach by Green and Naghdi (1977, 1978), Day (1977) and Naghdi (1980). In essence, the criticisms stem from the inability of the rational thermodynamics version of the Clausius-Duhem inequality to rule out physically unacceptable behavior in some materials and an unclear relation between the second law and the Clausius-Duhem

inequality. Furthermore, as Day (1977) has noted, the rational thermodynamic approach affords no way of giving a unique meaning to the concept of entropy. As an example, consider the case of a rigid heat conductor in thermal equilibrium. If heat is added to the conductor, the form of the Clausius-Duhem inequality that follows from rational thermodynamics predicts that the subsequent spatially homogeneous temperature field of the conductor decreases, which is contradictory to physical experience (Green and Naghdi, 1977). The Onsagerist approach, on the other hand, does not suffer from the above criticisms and is generally accepted by the majority of researchers in the field. Accordingly, we adopt the Onsagerist approach as the basis for the present study. In the following section, the pertinent thermodynamic relations that follow from this approach are presented.

2.2 BASIC RELATIONS FROM ONSAGERIST APPROACH.

It is well-known that the state of a reversible system can be completely described by functions which depend only on observable state variables. An irreversible system, on the other hand, is more complicated and, to define its state, requires state functions that depend not only on observable state variables but also on internal (non-observable) variables $\underset{\sim}{\xi}$ which describe the internal changes that occur in a material during an irreversible process. Reversible processes can then be treated within the framework of irreversible thermodynamics as processes for which the internal variables $\underset{\sim}{\xi}$ remain fixed. In this section, the basic relations of the Onsagerist school of irreversible thermodynamics with internal variables are presented; the discussion is modeled on that given by Nemat-Nasser (1975,1978).

The first law of thermodynamics takes the following rate form

$$\dot{Q} = \dot{e} - \underset{\sim}{\sigma} \cdot \dot{\underset{\sim}{\varepsilon}} \tag{2.1}$$

for incremental processes, regardless of whether they are reversible or irreversible. In (2.1) $(\dot{})$ denotes an appropriate objective rate of $()$. Here, Q is the heat added per unit mass, e denotes the internal energy per unit mass, $\underset{\sim}{\sigma}$ represents the stress tensor, and $\underset{\sim}{\varepsilon}$ is the small strain tensor.

The second law of thermodynamics requires the definition of entropy and an expression for entropy production. For the case of reversible systems, the existence of entropy was first proven by Carathéodory (see Buchdahl, 1966). Shortly thereafter,

Valanis (1971), using a physical argument, addressed the question of entropy for irreversible systems and, in a more recent work (1983), he formally proved the existence of entropy for such systems mathematically. Using the concept of partial integrability of Pfaffian forms, Valanis finds the temperature $\theta = \theta(\underset{\sim}{\xi}, e, \underset{\sim}{\xi}) > 0$ as an integrating factor of Eq. (2.1), which renders $\mathring{Q}/\theta = \dot{\eta}$ as an exact differential form for $\underset{\sim}{\xi} = \underset{\sim}{\xi}_o$ (= constant), where η is called the entropy:

$$\eta = \eta(\underset{\sim}{\xi}, e, \underset{\sim}{\xi}) \quad . \tag{2.2}$$

For the case $\underset{\sim}{\xi} = \underset{\sim}{\xi}_o$, the differential form of Eq. (2.2) and Eq. (2.1) yields

$$\dot{\eta} = \frac{\partial \eta}{\partial e} \dot{e} + \frac{\partial \eta}{\partial \underset{\sim}{\xi}} \cdot \dot{\underset{\sim}{\xi}} = \frac{1}{\theta} \dot{e} - \frac{1}{\theta} \underset{\sim}{q} \cdot \dot{\underset{\sim}{\xi}} \ , \tag{2.3}$$

from which it follows that

$$\underset{\sim}{q} = - \theta \frac{\partial \eta}{\partial \underset{\sim}{\xi}} \quad , \quad \frac{1}{\theta} = \frac{\partial \eta}{\partial e} \quad . \tag{2.4}$$

For an arbitrary irreversible process for which $\dot{\underset{\sim}{\xi}} \neq 0$, Eq. (2.2) together with Eqs. (2.4) leads to the expression:

$$\begin{aligned} \dot{\eta} &= \frac{\partial \eta}{\partial \underset{\sim}{\xi}} \cdot \dot{\underset{\sim}{\xi}} + \frac{\partial \eta}{\partial e} \dot{e} + \frac{\partial \eta}{\partial \underset{\sim}{\xi}} \cdot \dot{\underset{\sim}{\xi}} \\ &= \frac{1}{\theta} \left(\dot{e} - \underset{\sim}{q} \cdot \dot{\underset{\sim}{\xi}} \right) + \frac{\partial \eta}{\partial \underset{\sim}{\xi}} \cdot \dot{\underset{\sim}{\xi}} = \frac{\mathring{Q}}{\theta} + \frac{\partial \eta}{\partial \underset{\sim}{\xi}} \cdot \dot{\underset{\sim}{\xi}} \quad . \end{aligned} \tag{2.5}$$

Upon decomposing $\dot{\eta}$ into a reversible part $\dot{\eta}_r$ and a dissipative part $\dot{\eta}_d$, Eq. (2.5) can be rewritten in the form:

$$\theta \dot{\eta} = \theta \dot{\eta}_r + \theta \dot{\eta}_d \tag{2.6}$$

where

$$\theta \dot{\eta}_r = \mathring{Q} \quad , \quad \theta \dot{\eta}_d = \theta \frac{\partial \eta}{\partial \underset{\sim}{\xi}} \cdot \dot{\underset{\sim}{\xi}} \quad . \tag{2.7}$$

The second law of thermodynamics states that the rate of irreversible entropy production $\dot{\eta}_d$ must be either zero (for reversible processes) or positive (for irreversible processes). Thus, we must have

$$\mathcal{D} \equiv \theta \dot{\eta}_d = \theta \frac{\partial \eta}{\partial \underset{\sim}{\xi}} \cdot \dot{\underset{\sim}{\xi}} \geq 0 \tag{2.8}$$

Alternately, the state variables e and η can be expressed in terms of $\underset{\sim}{\xi}$, θ and $\underset{\sim}{\xi}$ as follows:

$$e = \hat{e}\left(\underset{\sim}{\xi}, \theta, \underset{\sim}{\xi}\right) \quad , \quad \eta = \eta\left(\underset{\sim}{\xi}, \hat{e}, \underset{\sim}{\xi}\right) = \hat{\eta}\left(\underset{\sim}{\xi}, \theta, \underset{\sim}{\xi}\right) \quad . \tag{2.9}$$

Then, for the case in which $\underset{\sim}{\xi} = \underset{\sim}{\xi}_o$, Eq. (2.5) can be rewritten as

$$\left.\begin{aligned} \dot{\eta} &= \frac{\partial \eta}{\partial \underset{\sim}{\xi}} \cdot \dot{\underset{\sim}{\xi}} + \frac{\partial \eta}{\partial \theta} \dot{\theta} = \frac{1}{\theta} \left[\frac{\partial \hat{e}}{\partial \underset{\sim}{\xi}} \cdot \dot{\underset{\sim}{\xi}} + \frac{\partial \hat{e}}{\partial \theta} \dot{\theta} \right] - \frac{1}{\theta} \underset{\sim}{q} \cdot \dot{\underset{\sim}{\xi}} \\ &= \frac{1}{\theta} \left[\frac{\partial \hat{e}}{\partial \underset{\sim}{\xi}} - \underset{\sim}{q} \right] \cdot \dot{\underset{\sim}{\xi}} + \frac{1}{\theta} \frac{\partial \hat{e}}{\partial \theta} \dot{\theta} \quad . \end{aligned}\right\} \tag{2.10}$$

From Eq. (2.10), it follows that

$$\frac{\partial \hat{\eta}}{\partial \underset{\sim}{\xi}} = \frac{1}{\theta} \left[\frac{\partial \hat{e}}{\partial \underset{\sim}{\xi}} - \underset{\sim}{q} \right] \quad , \tag{2.11a}$$

$$\frac{\partial \hat{\eta}}{\partial \theta} = \frac{1}{\theta} \frac{\partial \hat{e}}{\partial \theta} \quad . \tag{2.11b}$$

The Helmholtz free energy ψ is now introduced, where

$$\psi\left(\underset{\sim}{\xi}, \theta, \underset{\sim}{\xi}\right) \equiv \hat{e} - \theta \hat{\eta} \quad . \tag{2.12}$$

From Eq. (2.11a) it follows that

$$\underset{\sim}{q} = \frac{\partial \psi}{\partial \underset{\sim}{\xi}} \quad . \tag{2.13}$$

Furthermore, by taking the derivative of Eq. (2.12) with respect to θ and using Eq. (2.11b), one obtains the result

$$\eta = - \frac{\partial \psi}{\partial \theta} \quad . \tag{2.14}$$

In order to express Eq. (2.8) in terms of the variables, ϵ, θ and ξ, let us write

$$\begin{aligned} \frac{\partial}{\partial \xi} \hat{\eta} \ (\xi, \theta, \xi) &= \frac{\partial}{\partial \xi} \eta \left(\xi, \hat{e}(\xi, \theta, \xi), \xi\right) \\ &= \frac{\partial \eta}{\partial \xi} + \frac{\partial \eta}{\partial \hat{e}} \cdot \frac{\partial \hat{e}}{\partial \xi} \quad . \end{aligned} \tag{2.15}$$

Upon combining Eqws. (2.4b) and (2.15), it folows that

$$\theta \frac{\partial \eta}{\partial \xi} = - \frac{\partial \hat{e}}{\partial \xi} + \theta \frac{\partial \hat{\eta}}{\partial \xi} = - \frac{\partial \psi}{\partial \xi} \quad . \tag{2.16}$$

As a result, the second law inequality (2.8) can be placed in the form

$$\overset{\bullet}{D} = \underset{\sim}{A} \cdot \overset{\bullet}{\xi} \geq 0 \tag{2.17}$$

where the internal force, $\underset{\sim}{A}$, is defined by

$$\underset{\sim}{A} = - \frac{\partial \psi}{\partial \xi} \quad . \tag{2.18}$$

In a formulation based upon taking ϱ, θ and ξ as the independent variables, Eq. (2.13) may be solved to give

$$\xi = \hat{\xi}(\varrho, \theta, \xi) \quad . \tag{2.19}$$

Upon introducing the Legendre transformation:

$$\Phi = \varrho : \xi - \psi \quad , \tag{2.20}$$

it follows from Eq. (2.20) that

$$\underset{\sim}{\xi} = \frac{\partial \phi}{\partial \underset{\sim}{q}} \quad , \tag{2.21}$$

$$\eta = \frac{\partial \phi}{\partial \theta} \quad , \tag{2.22}$$

$$\underset{\sim}{A} = \frac{\partial \phi}{\partial \underset{\sim}{\xi}} \quad . \tag{2.23}$$

The rate-form of Eq. (2.21), with the strain-rate decomposition:

$$\dot{\underset{\sim}{\xi}} = \dot{\underset{\sim}{\xi}}_e + \dot{\underset{\sim}{\xi}}_p + \dot{\underset{\sim}{\xi}}_\theta \ , \tag{2.24}$$

leads to the following relations

$$\dot{\underset{\sim}{\xi}}_e = \frac{\partial^2 \phi}{\partial \underset{\sim}{q} \partial \underset{\sim}{q}} \dot{\underset{\sim}{q}} \ , \quad \dot{\underset{\sim}{\xi}}_p = \frac{\partial^2 \phi}{\partial \underset{\sim}{q} \partial \underset{\sim}{\xi}} \dot{\underset{\sim}{\xi}} \quad , \quad \dot{\underset{\sim}{\xi}}_\theta = \frac{\partial^2 \phi}{\partial \underset{\sim}{q} \partial \theta} \tag{2.25}$$

where the subscripts e, p, and θ, respectively, refer to the elastic, plastic and thermal components of the total strain rate.

Note that in Eq. (2.25a), $\partial^2 \phi / \partial \underset{\sim}{q} \partial \underset{\sim}{q}$ is a tangent compliance matrix.

2.3 CONDITIONS FOR THE EXISTENCE OF A PLASTIC POTENTIAL.

Non-associated plasticity rests on the basic premise that the plastic strain rate $\dot{\underset{\sim}{\xi}}_p$ is defined in terms of a scalar plastic potential function ϕ as follows:

$$\dot{\underset{\sim}{\xi}}_p = \lambda \frac{\partial \phi}{\partial \underset{\sim}{q}} \quad , \tag{2.26}$$

where the scalar λ is known as the plastic multiplier. We will now explore, within the framework of the irreversible thermodynamics just described, the conditions under which such a relationship exists.

The evolution equations for the internal variables $\underset{\sim}{\xi}$ are commonly taken in the general form:

$$\dot{\underset{\sim}{\xi}} = \underset{\sim}{H}\left(\underset{\sim}{A}, \underset{\sim}{\xi}\right) , \tag{2.27}$$

which implies that the rate at which any specific structural rearrangement occurs is fully determined by the current values of the associated generalized force $\underset{\sim}{A}$ and $\underset{\sim}{\xi}$. The influence of the macroscopic stress state on $\dot{\underset{\sim}{\xi}}$ appears only implicitly through the dependence of $\underset{\sim}{A}$ on $\underset{\sim}{\sigma}$, which can only be determined from micromechanical considerations relating the generalized force $\underset{\sim}{A}$ to the macroscopic stress $\underset{\sim}{\sigma}$ (see Rice, 1971).

In order to prove the existence of a plastic potential function ϕ, it is necessary to assume that the function $\underset{\sim}{H}$ is derivable from a scalar potential $\hat{\phi}(\underset{\sim}{A}, \underset{\sim}{\xi})$, such that:

$$\dot{\underset{\sim}{\xi}} = \underset{\sim}{H} = \lambda \frac{\partial \hat{\phi}}{\partial \underset{\sim}{A}} \tag{2.28}$$

where

$$\hat{\phi}\left(\underset{\sim}{A}, \underset{\sim}{\xi}\right) \equiv \int_0^{\underset{\sim}{A}} \frac{\underset{\sim}{H} \cdot d\underset{\sim}{A}}{\lambda} \tag{2.29}$$

and λ is an integrating denominator which is chosen to be positive, i.e.,

$$\lambda\left(\underset{\sim}{A}, \underset{\sim}{\xi}\right) > 0 \qquad \text{for } \dot{\underset{\sim}{\xi}} \neq 0 \quad . \tag{2.30}$$

From Eq. (2.25), we can write

$$\dot{\underset{\sim}{\varepsilon}}_p = \frac{\partial^2 \phi}{\partial \underset{\sim}{\sigma} \partial \underset{\sim}{\xi}} \dot{\underset{\sim}{\xi}} , \tag{2.31}$$

which can be combined with Eq. (2.28) to yield

$$\dot{\underset{\sim}{\varepsilon}}_p = \lambda \frac{\partial \underset{\sim}{A}}{\partial \underset{\sim}{\sigma}} \frac{\partial \hat{\phi}}{\partial \underset{\sim}{A}} = \lambda \frac{\partial \hat{\phi}}{\partial \underset{\sim}{\sigma}} , \tag{2.32}$$

since

$$\underset{\sim}{A} = \frac{\partial \phi}{\partial \underset{\sim}{\xi}} . \tag{2.33}$$

With a change of variable, we can write*

$$\hat{\phi}(\underset{\sim}{A},\underset{\sim}{\xi}) = \phi(\underset{\sim}{\sigma},\underset{\sim}{\xi}) \quad , \tag{2.34}$$

so that Eq. (2.32) can be written as

$$\dot{\underset{\sim}{\xi}}_p = \lambda \frac{\partial \phi}{\partial \underset{\sim}{\sigma}} \quad , \tag{2.35}$$

which is identical to Eq. (2.26) and shows that $\phi(\underset{\sim}{\sigma},\underset{\sim}{\xi})$ is the plastic flow potential with λ the plastic multiplier. To obtain this expression, it was necessary to assume that Eq. (2.28) holds. In that which follows, we examine the conditions under which this assumption is valid.

It has been pointed out (Nemat-Nasser, 1978) that, due to the second law, expressed by Eq. (2.17), $\dot{\underset{\sim}{\xi}}$ cannot be prescribed arbitrarily in the same manner as $\dot{\underset{\sim}{\sigma}}$ or $\dot{\underset{\sim}{\varepsilon}}$. Also, the second law does not restrict the sign of the term $\dot{\underset{\sim}{A}} \cdot \dot{\underset{\sim}{\xi}}$. Based on the sign of this term, a stability concept has been introduced (Valanis, 1968, 1971c; Nemat-Nasser, 1978). A *material is said to be thermodynamically stable if:***

$$\dot{\underset{\sim}{A}} \cdot \dot{\underset{\sim}{\xi}} < 0 \tag{2.36}$$

and otherwise is thermodynamically unstable. For stable materials, $\underset{\sim}{A}$ relaxes when $\dot{\underset{\sim}{\xi}} = 0$, while for unstable materials, $\underset{\sim}{A}$ increases when $\dot{\underset{\sim}{\xi}} = 0$. On the basis of physical considerations, thermodynamically unstable materials are excluded from further discussion.

Restricting attention now to thermodynamically stable materials, let us consider the conditions that must be imposed on the function $\underset{\sim}{H}$ to insure the existence of a potential $\hat{\phi}$ (or ϕ). Specifically, it can be shown (see Nemat-Nasser, 1975a; 1978) that $\hat{\phi}$ (or ϕ) will exist if either one of the following two conditions are satisfied:

(i) The generalized Onsager relation holds

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* Again, this relationship can be defined explicitly only on the basis of micromechanical considerations.

** This concept of stability is not related to the stability postulate of Drucker (1954).

$$\frac{\partial}{\partial \Lambda_i}\left(\frac{H_j}{\lambda}\right) = \frac{\partial}{\partial \Lambda_j}\left(\frac{H_i}{\lambda}\right) \quad , \tag{2.37}$$

where λ is an integrating denominator;

(ii) the function $\underset{\sim}{H}$ depends only on $\underset{\sim}{\Lambda}$, i.e.,

$$\underset{\sim}{H} = \underset{\sim}{H}(\underset{\sim}{\Lambda}) \quad , \tag{2.38}$$

and the number of internal variables ξ_i is less than 8.

Condition (i) is satisfied, for example, when $\underset{\sim}{H}$ depends linearly on $\underset{\sim}{\lambda}$, i.e.,

$$H_i = B_{ij}\, \Lambda_j \quad , \tag{2.39}$$

where B_{ij} represents a symmetric positive definite matrix to satisfy the second law inequality (2.17). Condition (ii), on the other hand, is satisfied when

$$\underset{\sim}{H} = \dot{\underset{\sim}{\varepsilon}}_p \quad . \tag{2.40}$$

Again, the validity of the above assumptions can be judged only on the basis of micromechanical considerations of the response of the particular material under consideration which is outside of the scope of this report.

2.4 RESTRICTIONS IMPOSED BY THE SECOND LAW.

There are basically two situations for which strong statements can be made about the restrictions imposed by the second law on non-associated flow rules. In the first situation, the potential function $\hat{\phi}$ must be a homogeneous function of the generalized force $\underset{\sim}{\Lambda}$, while in the second situation the Helmholtz free energy function ψ must be prescribed, together with evolutionary equations for the internal variables.

Considering now the first situation, we can write from Eqs. (2.17) and (2.28) with (2.34) that

$$\mathring{D} = \underset{\sim}{\Lambda} \cdot \lambda \frac{\partial \hat{\phi}}{\partial \underset{\sim}{\Lambda}} = n\lambda \hat{\phi}(\underset{\sim}{\Lambda}, \xi) = n\lambda \phi(\underset{\sim}{\sigma}, \xi) \geq 0 \ . \tag{2.41}$$

Since $n > 0$, the second law requires only that $\lambda\phi(\underset{\sim}{\sigma},\xi) \geq 0$. If, as is customary, ϕ is taken to be positive, the the second law requires that

$$\lambda > 0 \ , \tag{2.42}$$

which is consistent with Eq. (2.30) given earlier. Thus, for a positive potential function ϕ which is homogeneous in $\underset{\sim}{\Lambda}$, Eq. (2.42) expresses the only restriction imposed by the second law. Inequality (2.42) is often used to distinguish plastic loading from unloading when the stress state is on the yield surface.

Consider now a special subclass of the above materials for which $\hat{\phi}$ is a *homogeneous function of* $\underset{\sim}{\Lambda}$, *and* $\underset{\sim}{\Lambda}$ *is related to* $\underset{\sim}{\sigma}$ *through a relation*

$$\underset{\sim}{\Lambda} = \underset{\sim}{\Lambda}(\underset{\sim}{\sigma} - \underset{\sim}{\sigma}_o, \xi) \ , \tag{2.43}$$

such that

$$\phi[\underset{\sim}{\Lambda}(\underset{\sim}{\sigma} - \underset{\sim}{\sigma}_o, \xi)] = \phi(\underset{\sim}{\sigma} - \underset{\sim}{\sigma}_o, \xi) , \tag{2.44}$$

and ϕ is a homogeneous function of degree m (>o) of the argument $\underset{\sim}{\sigma} - \underset{\sim}{\sigma}_o$. Here, $\underset{\sim}{\sigma}_o$ is called the back stress and generally depends on ξ. This subclass includes a wide variety of engineering materials whose plastic yield characteristics exhibit both kinematic and isotropic hardening. For these materials, the plastic potential can be expressed in the form:

$$\hat{\phi}(\underset{\sim}{\Lambda}, \xi) = \phi(\underset{\sim}{\sigma}, \xi) = \phi_o[\underset{\sim}{\sigma} - \underset{\sim}{\sigma}_o(\xi)] + \phi_1[\xi] \ . \tag{2.45}$$

Here, ϕ_o describes kinematic hardening. Since ϕ_o is assumed to be a homogeneous function of degree m(>0) of the argument $\underset{\sim}{\sigma} - \underset{\sim}{\sigma}_o(\xi)$, the same will be true for ϕ_o. With this in mind, it follows that

$$\dot{\underset{\sim}{\varepsilon}}_p = \lambda \frac{\partial \phi}{\partial \underset{\sim}{\sigma}} = \lambda \frac{\partial}{\partial \underset{\sim}{\sigma}} \phi_o(\underset{\sim}{\sigma} - \underset{\sim}{\sigma}_o) \quad . \tag{2.46}$$

As a result, the rate of plastic work, $\dot{W}_p$, can be extressed as

$$\dot{W}_p = (\underset{\sim}{\sigma} - \underset{\sim}{\sigma}_o) \cdot \dot{\underset{\sim}{\varepsilon}}_p = \lambda \frac{\partial \phi_o}{\partial \underset{\sim}{\sigma}} (\underset{\sim}{\sigma} - \underset{\sim}{\sigma}_o) = \lambda m \phi_o \quad , \tag{2.47}$$

so that, for a positive plastic potential and $\lambda > 0$, it follows that

$$\dot{W}_p > 0 \quad \text{for} \quad \dot{\underset{\sim}{\varepsilon}}_p \neq 0 \quad . \tag{2.48}$$

Thus, the second law requires positive plastic work for those materials whose plastic potentials are of the form of Eq. (2.45).

In the second situation mentioned above, the Helmholtz free energy function ψ and the evolutionary equation for $\dot{\xi}$ are prescribed. In this case, the second law, as reflected by Eq. (2.17), imposes the following condition:

$$- \frac{\partial \psi}{\partial \xi} \dot{\xi} \geq 0 \quad , \tag{2.49}$$

which places restrictions on the forms that ψ and $\dot{\xi}$ may take. This approach was taken by Valanis (1971) in developing the endochronic theory. For this purpose, he postulated the simplest isotropic forms for the Helmholtz function ψ and the evolutionary equations for $\dot{\xi}$ which satisfy Eq. (2.49). With these, he was able to integrate the evolutionary equations and subsequently eliminate the internal variables from the stress-strain relations. In later work, Valanis (1980) showed that this approach can be used to derive the equations of classical associated plasticity with both kinematic and isotropic hardening.

In summary, it follows from the discussion above that, unless the Helmholtz free energy function ψ and the evolutionary equations for $\dot{\xi}$ are known, the only restriction imposed by the second law on non-associated flow rules is that the plastic multiplier λ must be positive. Since the general approach followed in practice is to postulate forms for the plastic potential function ϕ and the yield function f, and not to prescribe either ψ or $\dot{\xi}$, the restriction imposed by the second law, $\lambda > 0$, is very weak and offers little constraint to the non-associated flow rule.

SECTION 3

THE POSTULATES OF DRUCKER AND IL'IUSHIN

In the past, the postulates of Drucker (1960) and Il'iushin (1961) have been adopted as bases for developing certain plasticity theories and for assessing the acceptability of others, despite the fact that these postulates are not based on the laws of thermodynamics. The purpose of this section is to show that these postulates can be violated without causing a violation of the second law of thermodynamics. As a result, it follows that they should not be used as criteria to judge the thermodynamic acceptability of inelastic constitutive theories.

3.1 DRUCKER'S POSTULATES.

Consider an elasto-plastic material in equilibrium under a state of stress $\underset{\sim}{\sigma}^{o}$ and $\underset{\sim}{\varepsilon}^{o}$, denoted by state 0. An external agency slowly applies an independent set of forces in equilibrium, reaching state 1 with stress $\underset{\sim}{\sigma}'$ and strain $\underset{\sim}{\varepsilon}'$. Drucker's postulate of stability in the large requires that the following inequality hold:

$$\int_{\underset{\sim}{\varepsilon}^{o}}^{\underset{\sim}{\varepsilon}'} \left(\underset{\sim}{\sigma} - \underset{\sim}{\sigma}^{o}\right) \cdot d\underset{\sim}{\varepsilon} \geq 0 \tag{3.1}$$

Over a cycle of application and removal of the additional stress, it is also postulated that the net work performed by the external agency is non-negative:

$$\int_{\underset{\sim}{\varepsilon}_p^{o}}^{\underset{\sim}{\varepsilon}_p'} \left(\sigma - \underset{\sim}{\sigma}^{o}\right) \cdot d\underset{\sim}{\varepsilon}_p \geq 0 \quad , \tag{3.2}$$

where $\underset{\sim}{\varepsilon}_p^{o}$ and $\underset{\sim}{\varepsilon}_p'$ denote the plastic strains at states 0 and 1, respectively. If these states are an infinitesimal distance apart, we can write $\underset{\sim}{\sigma}' - \underset{\sim}{\sigma}^{o} = \Delta\underset{\sim}{\sigma}$ and Eqs. (3.1) and (3.2) reduce to the expressions:

$$\Delta \underset{\sim}{\sigma} \cdot \Delta \underset{\sim}{\varepsilon} \geq 0 \quad ,$$

$$\Delta \underset{\sim}{\sigma} \cdot \Delta \underset{\sim}{\varepsilon}_p \geq 0 \quad . \tag{3.3}$$

From Eq. (2.24) in the preceding section, we have for isothermal processes:

$$\Delta \underset{\sim}{\varepsilon} = \Delta \underset{\sim}{\varepsilon}_e + \Delta \underset{\sim}{\varepsilon}_p = \frac{\partial^2 \phi}{\partial \underset{\sim}{\sigma} \partial \underset{\sim}{\sigma}} \Delta \underset{\sim}{\sigma} + \frac{\partial^2 \phi}{\partial \xi \partial \underset{\sim}{\sigma}} \Delta \xi \quad . \tag{3.4}$$

Since the elastic compliance is positive definite, it follows that

$$\Delta \underset{\sim}{\sigma} \cdot \Delta \underset{\sim}{\varepsilon}_e = \Delta \underset{\sim}{\sigma} \cdot \frac{\partial^2 \phi}{\partial \underset{\sim}{\sigma} \partial \underset{\sim}{\sigma}} \Delta \underset{\sim}{\sigma} > 0 \quad \text{for } \Delta \underset{\sim}{\sigma} \neq \underset{\sim}{0} \quad . \tag{3.5}$$

Inasmuch as the inequalities (3.3b) and (3.5) are sufficient conditions for Eq. (3.3a), only the inequality (3.3b) needs to be considered. Using Eq. (3.4) we write

$$\Delta \underset{\sim}{\sigma} \cdot \Delta \underset{\sim}{\varepsilon}_p = \Delta \underset{\sim}{\sigma} \cdot \frac{\partial^2 \phi}{\partial \xi \partial \underset{\sim}{\sigma}} \Delta \xi \quad , \tag{3.6}$$

which, in view of Eqs. (2.17) and (2.23), can be rewritten in the form:

$$\Delta \underset{\sim}{\sigma} \cdot \Delta \underset{\sim}{\varepsilon}_p = \Delta \underset{\sim}{\sigma} \cdot \frac{\partial (\Delta D)}{\partial \underset{\sim}{\sigma}} \geq 0 \quad . \tag{3.7}$$

This equation expresses Drucker's postulate of stability in the small in terms of the increment of dissipation, ΔD. Since the second Law requires only that $\Delta D \geq 0$, it follows that the inequality (3.7) can be violated for processes in which the second law is not violated.

3.2 IL'IUSHIN'S POSTULATE.

A postulate introduced by Il'iushin (1961) states that if the work ΔW done by external forces in an isothermal closed cycle of strain is positive, plastic deformation takes place and, if the work ΔW is zero, only elastic deformation occurs. Expressed mathematically, this postulate takes the form:

$$\Delta W = \oint_{\underset{\sim}{\varepsilon}} \underset{\sim}{\sigma} \cdot d\underset{\sim}{\varepsilon} \geq 0 \quad , \tag{3.8}$$

where the integration is taken around a closed path in strain space. Since a strain cycle is always possible, even for softening stress-strain behavior, Il'iushin's postulate is less restrictive than Drucker's postulate.***

Consider a small strain cycle, such that $\underset{\sim}{\varepsilon}_o \rightarrow \underset{\sim}{\varepsilon}_1 = \underset{\sim}{\varepsilon}_o + \Delta\underset{\sim}{\varepsilon} \rightarrow \underset{\sim}{\varepsilon}_2 = \underset{\sim}{\varepsilon}_o$, during which the stress changes as follows: $\underset{\sim}{\sigma}_o \rightarrow \underset{\sim}{\sigma}_1 \rightarrow \underset{\sim}{\sigma}_2$. The work done during this cycle is given by the expression:

$$\Delta W = \int_{\underset{\sim}{\varepsilon}_o}^{\underset{\sim}{\varepsilon}_o+\Delta\underset{\sim}{\varepsilon}} \underset{\sim}{\sigma} \cdot d\underset{\sim}{\varepsilon} + \int_{\underset{\sim}{\varepsilon}_o+\Delta\underset{\sim}{\varepsilon}}^{\underset{\sim}{\varepsilon}_o} \underset{\sim}{\sigma} \cdot d\underset{\sim}{\varepsilon} = \frac{1}{2}\left(\underset{\sim}{\sigma}_o + \underset{\sim}{\sigma}_1\right) \cdot \Delta\underset{\sim}{\varepsilon} - \frac{1}{2}\left(\underset{\sim}{\sigma}_1 + \underset{\sim}{\sigma}_2\right) \cdot \Delta\epsilon \quad . \tag{3.9}$$

Hence

$$\Delta W = \frac{1}{2}\left(\underset{\sim}{\sigma}_o - \underset{\sim}{\sigma}_2\right) \cdot \Delta\underset{\sim}{\varepsilon} \quad . \tag{3.10}$$

Let us now assume that irreversible changes occur only during loading. Then the internal variables ξ will evolve as follows:

$$\xi_o \rightarrow \xi_1 = \xi_o + \Delta\xi \rightarrow \xi_1 \quad . \tag{3.11}$$

From Eq. (2.13), we can write

$$\underset{\sim}{\sigma}_o - \underset{\sim}{\sigma}_2 = \frac{\partial}{\partial\underset{\sim}{\varepsilon}} \Psi(\underset{\sim}{\varepsilon}_o, \xi_o) - \frac{\partial}{\partial\underset{\sim}{\varepsilon}} \Psi(\underset{\sim}{\varepsilon}_o, \xi_o + \Delta\xi) \quad , \tag{3.12}$$

*** Drucker's postulate expressed by the inequality (3.2) requires a stress cycle. Such a cycle is not possible on the softening branch for materials with softening stress-strain relations.

so that

$$\underset{\sim}{\ell}_o - \underset{\sim}{\ell}_2 = - \frac{\partial^2 \psi}{\partial \underset{\sim}{\xi} \partial \underset{\sim}{\xi}} \cdot \Delta \underset{\sim}{\xi} \quad . \tag{3.13}$$

Thus, in view of Eqs. (3.8), (3.10), and (3.13), we can write

$$\frac{\partial}{\partial \underset{\sim}{\xi}} (\Delta D) \cdot \Delta \underset{\sim}{\xi} \geq 0 \quad , \tag{3.14}$$

which expresses Il'iushin's postulate in terms of the gradient of the dissipation increment, $\partial(\Delta D)/\partial \underset{\sim}{\xi}$, and the amplitude of the strain cycle, $\Delta \underset{\sim}{\xi}$. Since the second law requires only that $\Delta D \geq 0$, it follows that the inequality (3.14) can be violated without violating the second law. Thus, the work done in a closed strain cycle may be negative without violating the laws of thermodynamics.

3.3 CLOSURE.

In this section, we have expressed both Drucker's and Il'iushin's postulates in terms of the dissipation increment, ΔD, so that a direct connection with the second law can be explored. This appears to be a new result. An examination of the resulting expressions clearly shows that both postulates are not related to the second law of thermodynamics.

SECTION 4

NON-ASSOCIATED FLOW LAWS

In the preceding section, the restrictions imposed by thermodynamics on elasto-plastic constitutive theories, in general, were established; these restrictions apply to both associated and non-associated flow laws. In this section, attention is specifically focused on non-associated flow laws, and the implications of the thermodynamic restrictions for such laws are explored. In addition, an unusual feature of non-associated plasticity is discussed, and it is shown that non-associated flow laws satisfy the continuity condition of Prager (1949). Finally, path-dependence of non-associated plasticity is considered.

4.1 RESTRICTION IMPOSED BY THE SECOND LAW.

Consider an elastic-plastic material having a yield criterion of the form:

$$f(\underset{\sim}{\sigma},\kappa) = 0 \tag{4.1}$$

where κ is some scalar hardening parameter. The elastic portion of the deformation is governed by an incremental form of Hooke's Law, which we write as:

$$\dot{\underset{\sim}{\sigma}} = \underset{\sim}{\underset{\sim}{L}}\left(\dot{\underset{\sim}{\varepsilon}} - \dot{\underset{\sim}{\varepsilon}}_p\right) \tag{4.2}$$

where L is the elastic modulus tensor. The plastic strain rate $\dot{\underset{\sim}{\varepsilon}}_p$ is taken to follow the non-associated flow law:

$$\dot{\underset{\sim}{\varepsilon}}_p = \lambda \frac{\partial \phi}{\partial \underset{\sim}{\sigma}} \tag{4.3}$$

where ϕ denotes the plastic potential function. The difficulty of not being able to eliminate ξ in Eq. (2.30) in terms of observable variables for $\dot{\xi} \neq 0$ due to the lack of the information on the microstructural irreversible processes has been circumvented by the use of Prager's consistency condition. The consistency condition for the yield surface $f(\underset{\sim}{\sigma},\kappa)$, with plastic work hardening given by $\dot{\kappa} = \underset{\sim}{\sigma} \cdot \dot{\underset{\sim}{\varepsilon}}_p$, leads to the result

$$\lambda = \frac{\frac{\partial f}{\partial \underset{\sim}{q}} \cdot \underset{\approx}{L} \cdot \dot{\underset{\sim}{\varepsilon}}}{\frac{\partial f}{\partial \underset{\sim}{q}} \cdot \underset{\approx}{L} \cdot \frac{\partial \phi}{\partial \underset{\sim}{q}} - \frac{\partial f}{\partial \kappa} \underset{\sim}{g} \cdot \frac{\partial \phi}{\partial \underset{\sim}{q}}} \quad . \tag{4.4}$$

It is straightforward to show that the numerator of Eq. (4.4) is positive for plastic loading and negative for elastic unloading. Therefore, in order to satisfy the general restriction imposed by the second law of thermodynamics, namely, $\lambda \geq 0$, it is necessary that

$$\frac{\partial f}{\partial \underset{\sim}{q}} \cdot \underset{\approx}{L} \cdot \frac{\partial \phi}{\partial \underset{\sim}{q}} - \frac{\partial f}{\partial \kappa} \underset{\sim}{g} \cdot \frac{\partial \phi}{\partial \underset{\sim}{q}} \geq 0 \quad , \tag{4.5}$$

since the denominator of Eq. (4.4) is a function of $\underset{\sim}{q}$ and κ and not of $\dot{\underset{\sim}{\varepsilon}}$ and $\dot{\underset{\sim}{q}}$. The inequality (4.5) provides the basic restriction imposed on non-associated plastic flow laws by thermodynamics for materials which exhibit plastic work hardening. When there is no hardening, i.e., perfect plasticity, the inequality (4.5) reduces to the condition

$$\frac{\partial f}{\partial \underset{\sim}{q}} \cdot \underset{\approx}{L} \cdot \frac{\partial \phi}{\partial \underset{\sim}{q}} \geq 0 \tag{4.6}$$

which was first proposed by Bleich (1972). We shall return to Eq. (4.6) later on in Section 6 to explore specific restrictions on constitutive models of particular interest.

4.2 THE CONTINUITY CONDITION.

The continuity condition, introduced by Prager (1949), is a general requirement for elasto-plastic constitutive theories which is designed to prohibit possible difficulties that might arise from the presence of awkward discontinuities in constitutive relations for neutral or near-neutral loading. Difficulties in using constitutive relations which violate the continuity condition have been reported in the past (Nelson and Baladi, 1977). In essence, the continuity condition requires that a neutral loading condition be a limiting case of either loading or unloading. As a result, the constitutive relations for loading and unloading must coincide for this limiting case so that there are no discontinuities in the constitutive response. The purpose of this section is to demonstrate that non-associated plastic flow laws do, in fact, satisfy the continuity condition.

Consider a work hardening material having a yield function in the form of Eq. (4.1), with

$$\kappa = \int \underset{\sim}{\alpha} \cdot d\underset{\sim}{\varepsilon}_p \quad . \tag{4.7}$$

From the consistency condition and Eq. (4.3), it follows that

$$\frac{\partial f}{\partial \underset{\sim}{\sigma}} \cdot \dot{\underset{\sim}{\sigma}} + \lambda \frac{\partial f}{\partial \kappa} \left(\underset{\sim}{\alpha} \cdot \frac{\partial \phi}{\partial \underset{\sim}{\sigma}} \right) = 0 \quad , \tag{4.8}$$

from which

$$\lambda = \frac{\frac{\partial f}{\partial \underset{\sim}{\sigma}} \cdot \dot{\underset{\sim}{\sigma}}}{-\frac{\partial f}{\partial \kappa} \underset{\sim}{\alpha} \cdot \frac{\partial \phi}{\partial \underset{\sim}{\sigma}}} \quad , \tag{4.9}$$

subject to the requirement that

$$-\frac{\partial f}{\partial \kappa} \underset{\sim}{\alpha} \cdot \frac{\partial \phi}{\partial \underset{\sim}{\sigma}} > 0 \quad . \tag{4.10}$$

Let us now rewrite Eq. 4.9) as

$$\lambda = H(\underset{\sim}{\sigma}, \kappa) \, || \dot{\underset{\sim}{\sigma}} || \, (\underset{\sim}{n}_f \cdot \underset{\sim}{n}_\sigma) \tag{4.11}$$

where

$$H(\underset{\sim}{\sigma}\ \kappa) = \frac{\left|\left|\frac{\partial f}{\partial \underset{\sim}{\sigma}}\right|\right|}{-\frac{\partial f}{\partial \kappa}\ \underset{\sim}{\sigma} \cdot \frac{\partial \phi}{\partial \underset{\sim}{\sigma}}} \quad ,$$

$$\underset{\sim}{n}_f \equiv \frac{\frac{\partial f}{\partial \underset{\sim}{\sigma}}}{\left|\left|\frac{\partial f}{\partial \underset{\sim}{\sigma}}\right|\right|} \quad , \tag{4.12}$$

$$\underset{\sim}{n}_\sigma = \frac{\dot{\underset{\sim}{\sigma}}}{\left|\left|\dot{\underset{\sim}{\sigma}}\right|\right|} \quad .$$

Substitution of Eq. (4.11) into Eq. (4.3) leads to the result

$$\dot{\underset{\sim}{\varepsilon}}_p = H(\underset{\sim}{\sigma},\kappa)\left|\left|\dot{\underset{\sim}{\sigma}}\right|\right|(\underset{\sim}{n}_f \cdot \underset{\sim}{n}_\sigma)\left|\left|\frac{\partial \phi}{\partial \underset{\sim}{\sigma}}\right|\right| \underset{\sim}{n}_\phi \tag{4.13}$$

where $\underset{\sim}{n}_\phi$ is defined by replacing f of (4.12b) with ϕ.

Taking the norm of Eq. (4.13), we obtain

$$\frac{\left|\left|\dot{\underset{\sim}{\varepsilon}}_p\right|\right|}{\left|\left|\dot{\underset{\sim}{\sigma}}\right|\right|} = H(\underset{\sim}{\sigma},\kappa)\left|\left|\frac{\partial \phi}{\partial \underset{\sim}{\sigma}}\right|\right| \cos\theta \tag{4.14}$$

where

$$\underset{\sim}{n}_f \cdot \underset{\sim}{n}_\sigma = \cos\theta \quad . \tag{4.15}$$

Equation (4.14) defines the manner in which the magnitude of the plastic strain rate vector, $||\dot{\underset{\sim}{\varepsilon}}_p||$, varies with the direction and magnitude of the stress rate vector. From this expression, we see that

$$\lim_{\theta \to \pm\pi/2} \left|\left|\dot{\underset{\sim}{\varepsilon}}_p\right|\right| = 0 \quad . \tag{4.16}$$

Hence, the non-associated flow law satisfies the continuity condition.

4.3 A PECULIAR FEATURE OF NON-ASSOCIATED FLOW LAWS.

Non-associated flow laws exhibit an unusual feature of response that is not shared by associated flow laws. In particular, for a range of loading directions in stress space, the non-associated flow law can lead to response which is stiffer than purely elastic response. In this section, we examine this feature of non-associated plasticity, following the approach originally taken by Mroz (1966).

Consider an arbitrary stress rate vector, $\dot{\underset{\sim}{\sigma}}$, and let $\dot{\epsilon}_\sigma$ denote the projection of the strain rate vector, $\dot{\underset{\sim}{\epsilon}}$, on $\dot{\underset{\sim}{\sigma}}$, i.e.,

$$\dot{\epsilon}_\sigma = \left|\left|\dot{\underset{\sim}{\epsilon}}\right|\right| (\underset{\sim}{n}_\sigma \cdot \underset{\sim}{n}_\epsilon) = \frac{\dot{\underset{\sim}{\sigma}} \cdot \dot{\underset{\sim}{\epsilon}}}{||\underset{\sim}{\sigma}||} \quad , \tag{4.17}$$

Define directional and elastic moduli as follows:

$$K_d = \frac{||\dot{\underset{\sim}{\sigma}}||}{\dot{\epsilon}_\sigma} \quad , \quad K_e = \frac{||\dot{\underset{\sim}{\sigma}}||}{\dot{\epsilon}^e_\sigma} \quad , \tag{4.18}$$

so that from Eqs. (4.17) and (4.18), we can write

$$K_d = \frac{\left|\left|\dot{\underset{\sim}{\sigma}}\right|\right|^2}{\dot{\underset{\sim}{\sigma}} \cdot \dot{\underset{\sim}{\epsilon}}} \qquad K_e = \frac{\left|\left|\dot{\underset{\sim}{\sigma}}\right|\right|^2}{\dot{\underset{\sim}{\sigma}} \cdot \dot{\underset{\sim}{\epsilon}}_e} \quad . \tag{4.19}$$

From the strain rate decomposition

$$\dot{\underset{\sim}{\epsilon}} = \dot{\underset{\sim}{\epsilon}}_e + \dot{\underset{\sim}{\epsilon}}_p \quad , \tag{4.20}$$

it follows that

$$\dot{\underset{\sim}{\sigma}} \cdot \dot{\underset{\sim}{\epsilon}} = \dot{\underset{\sim}{\sigma}} \cdot \dot{\underset{\sim}{\epsilon}}_e + \dot{\underset{\sim}{\sigma}} \cdot \dot{\underset{\sim}{\epsilon}}_p \quad . \tag{4.21}$$

Let us rewrite Eq. (4.13) in the following form

$$\dot{\underset{\sim}{\varepsilon}}_p = \frac{1}{K} \left|\left|\dot{\underset{\sim}{\sigma}}\right|\right| \underset{\sim}{n}_\phi \cos\theta \tag{4.22}$$

where

$$K \equiv \left\{ H(\underset{\sim}{\sigma}, \kappa) \left|\left| \frac{\partial \phi}{\partial \underset{\sim}{\sigma}} \right|\right| \right\}^{-1} \quad , \tag{4.23}$$

and

$$\cos\theta \equiv \underset{\sim}{n}_f \cdot \underset{\sim}{n}_\sigma \quad .$$

Substitution of Eqs. (4.19) and (4.22) into Eq. (4.21) results is

$$\frac{1}{K_d} \left|\left|\dot{\underset{\sim}{\sigma}}\right|\right|^2 = \frac{1}{K_e} \left|\left|\dot{\underset{\sim}{\sigma}}\right|\right|^2 + \frac{1}{K} \left|\left|\dot{\underset{\sim}{\sigma}}\right|\right|^2 (\underset{\sim}{n}_\phi \cdot \underset{\sim}{n}_\sigma)\cos\theta \quad . \tag{4.24}$$

From Figure 1, we can write

$$\underset{\sim}{n}_\phi \cdot \underset{\sim}{n}_\sigma = \cos(\theta + \beta) \quad . \tag{4.25}$$

Thus, Eq. (4.24) can be placed in the form:

$$\frac{1}{K_d} = \frac{1}{K_e} + \frac{1}{K} \cos(\theta + \beta)\cos\theta \quad . \tag{4.26}$$

Note that when $0 \leq |\theta| < \pi/2$ and $\pi/2 < |\theta + \beta| < \pi$, it follows from Eq. (4.26) that

$$K_d > K_e \quad . \tag{4.27}$$

A typical non-dimensional plot of Eq. (4.26) is given in Figure 2, where the dimensionless directional modulus, K_d/K_e, is shown as a function of θ for different values of β. As this figure reveals, the modulus K_d can exceed the corresponding elastic modulus K_e for a range of values of β. This feature of non-associated plasticity is not shared by associated flow laws.

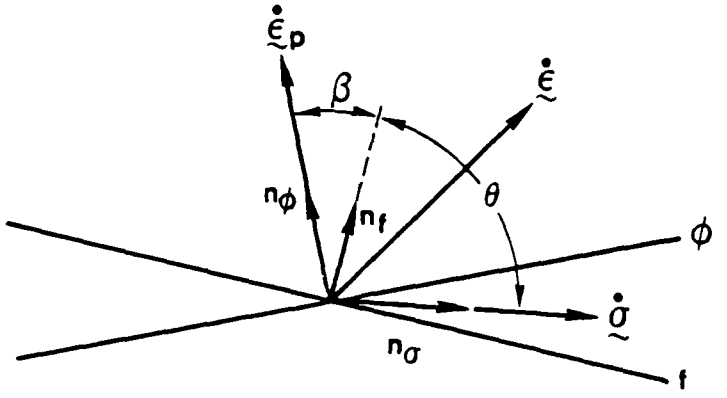

Figure 1. Geometry of the potential and yield surfaces at a point in stress space, showing the angles θ and β.

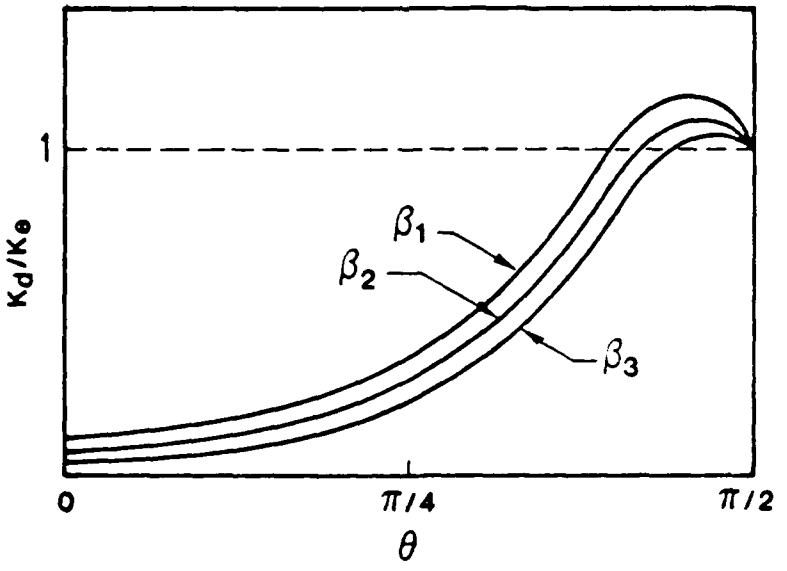

Figure 2. Typical features of Eq. (4.26) for various values of the parameter β.

4.4 PATH DEPENDENCE.

Another response feature of non-associated plasticity which is not found in associated plasticity is path-dependence. To show path dependence of non-associated plasticity, consider the following system of equations which define a general non-associated elasto-plastic material:

$$\dot{\underset{\sim}{\sigma}} = \underset{\sim}{\mathbb{L}} \left[\dot{\underset{\sim}{\varepsilon}} - \dot{\underset{\sim}{\varepsilon}}^{p} \right] \quad , \tag{4.28}$$

$$\dot{\underset{\sim}{\varepsilon}}^{p} = \lambda \frac{\partial \phi}{\partial \underset{\sim}{\sigma}} \quad , \tag{4.29}$$

$$f = f\left(\underset{\sim}{\sigma}, \kappa\right) \quad . \tag{4.30}$$

Here, Eq. (4.28) is the incremental form of Hooke's Law, with $\underset{\sim}{\mathbb{L}}$ the elastic modulus tensor; Eq. (4.29) is the non-associated flow law, with ϕ being the plastic potential. In addition, f denotes the yield function, which depends on $\underset{\sim}{\sigma}$ and a scalar hardening parameter, κ, where $\dot{\kappa} = \underset{\sim}{\sigma} \cdot \dot{\underset{\sim}{\varepsilon}}^{p}$.

From the consistency condition for $f(\underset{\sim}{\sigma},\kappa) = 0$, it follows that

$$\frac{\partial f}{\partial \underset{\sim}{\sigma}} \cdot \dot{\underset{\sim}{\sigma}} + \frac{\partial f}{\partial \kappa} \underset{\sim}{\sigma} \cdot \dot{\underset{\sim}{\varepsilon}}^{p} = 0 \quad . \tag{4.31}$$

By combining the above equations, it can be shown that

$$\lambda = \frac{\dfrac{\partial f}{\partial \underset{\sim}{\sigma}} \cdot \underset{\sim}{\mathbb{L}} \cdot \dot{\underset{\sim}{\varepsilon}}}{\dfrac{\partial f}{\partial \underset{\sim}{\sigma}} \cdot \underset{\sim}{\mathbb{L}} \cdot \dfrac{\partial \phi}{\partial \underset{\sim}{\sigma}} - \dfrac{\partial f}{\partial \kappa} \underset{\sim}{\sigma} \cdot \dfrac{\partial \phi}{\partial \underset{\sim}{\sigma}}} \quad . \tag{4.32}$$

which was given earlier in Eq. (4.4). Upon introducing Eq. (4.32) into Eq. (4.29), and substituting the result into Eq. (4.28), the following expression is obtained:

$$\dot{\underset{\sim}{\varrho}} = \left[\underset{\approx}{L} - \frac{\left(\underset{\approx}{L} \cdot \frac{\partial \phi}{\partial \underset{\sim}{\varrho}} \right) \left(\frac{\partial f}{\partial \underset{\sim}{\varrho}} \cdot \underset{\approx}{L} \right)}{\frac{\partial f}{\partial \underset{\sim}{\varrho}} \cdot \underset{\approx}{L} \cdot \frac{\partial \phi}{\partial \underset{\sim}{\varrho}} - \frac{\partial f}{\partial \kappa} \underset{\sim}{\varrho} \cdot \frac{\partial \phi}{\partial \underset{\sim}{\varrho}}} \right] \dot{\underset{\approx}{\epsilon}} \equiv \underset{\approx\approx}{B} \dot{\underset{\approx}{\epsilon}} \quad , \tag{4.33}$$

which can be written in the form:

$$\dot{\sigma}_{ij} = B_{ijk\ell} \dot{\epsilon}_{k\ell} \tag{4.34}$$

where

$$B_{ijk\ell} = L_{ijk\ell} - \frac{L_{ijmn} \frac{\partial \phi}{\partial \sigma_{mn}} \frac{\partial f}{\partial \sigma_{op}} L_{opk\ell}}{\tilde{H}(\varrho, \kappa)} \tag{4.35}$$

and

$$\tilde{H}(\varrho, \kappa) = \frac{\partial f}{\partial \underset{\sim}{\varrho}} \cdot \underset{\approx}{L} \cdot \frac{\partial \phi}{\partial \underset{\sim}{\varrho}} - \frac{\partial f}{\partial \kappa} \underset{\sim}{\varrho} \cdot \frac{\partial \phi}{\partial \underset{\sim}{\varrho}} \quad . \tag{4.36}$$

Now consider

$$B_{k\ell ij} = L_{k\ell ij} - \frac{L_{k\ell mn} \frac{\partial \phi}{\partial \sigma_{mn}} \frac{\partial f}{\partial \sigma_{op}} L_{opij}}{\tilde{H}(\varrho, \kappa)} \quad . \tag{4.37}$$

Since $L_{ijk\ell} = L_{k\ell ij}$, we can write Eq. (4.37) as

$$B_{k\ell ij} = L_{ijk\ell} - \frac{L_{ijop} \frac{\partial f}{\partial \sigma_{op}} \frac{\partial \phi}{\partial \sigma_{mn}} L_{mnk\ell}}{\tilde{H}(\varrho, \kappa)} \tag{4.38}$$

or, finally

$$B_{k\ell ij} = L_{ijk\ell} - \frac{L_{ijmn} \frac{\partial \phi}{\partial \sigma_{mn}} \frac{\partial f}{\partial \sigma_{op}} L_{opk\ell}}{\tilde{H}(\varrho, \kappa)} \quad . \tag{4.39}$$

A comparison of Eqs. (4.38) and (4.39) shows that, when $\phi \neq f$, we have

$$B_{k\ell ij} \neq B_{ijk\ell} \quad . \tag{4.40}$$

Consider now a rate-potential $\dot{W}$ defined such that

$$\dot{\sigma}_{ij} = \frac{\partial \dot{W}}{\partial \dot{\epsilon}_{ij}} \quad . \tag{4.41}$$

If such a potential exists, it follows that

$$\frac{\partial \dot{\sigma}_{ij}}{\partial \dot{\epsilon}_{k\ell}} = \frac{\partial^2 \dot{W}}{\partial \dot{\epsilon}_{k\ell} \partial \dot{\epsilon}_{ij}} = \frac{\partial^2 \dot{W}}{\partial \dot{\epsilon}_{ij} \partial \dot{\epsilon}_{k\ell}} = \frac{\partial \dot{\sigma}_{k\ell}}{\partial \dot{\epsilon}_{ij}} \tag{4.42}$$

which, in view of Eq. (4.34), implies that

$$B_{ijk\ell} = B_{k\ell ij} \quad . \tag{4.43}$$

Thus, since Eq. (4.43) holds for associated flow rules, a rate potential exists in this case. On the other hand, in view of Eq. (4.40), Eq. (4.42) is violated by non-associated flow rules and therefore a rate-potential does not exist for this case. This knowledge will be useful in the developments which follow.

Let us now consider the path-dependency of the strain energy function. For this purpose, we consider an incremental process from time t_o to $t_o + \Delta t$, and denote the stress and strain at time t_o by $\underset{\sim}{\sigma}_o$ and $\underset{\sim}{\epsilon}_o$. Over the time interval Δt, the increments in stress and strain are then given by the expressions:

$$\Delta \underset{\sim}{\sigma}_1 \equiv \int_{t_o}^{t_o+\Delta t} \dot{\underset{\sim}{\sigma}} dt \quad , \quad \Delta \underset{\sim}{\epsilon}_1 \equiv \int_{t_o}^{t_o+\Delta t} \dot{\underset{\sim}{\epsilon}} dt \quad . \tag{4.44}$$

The change in the strain energy during the incremental process is:

$$\Delta W = \int_{t_o}^{t_o+\Delta t} \underset{\sim}{\sigma} \cdot \dot{\underset{\sim}{\varepsilon}} dt = \int_{t_o}^{t_o+\Delta t} (\underset{\sim}{\sigma} - \underset{\sim}{\sigma}_o) \cdot \dot{\underset{\sim}{\varepsilon}} dt + \underset{\sim}{\sigma}_o \int_{t_o}^{t_o+\Delta t} \dot{\underset{\sim}{\varepsilon}} dt \quad , \tag{4.45}$$

which can be placed in the form

$$\Delta W = \int_0^{\Delta \underset{\sim}{\varepsilon}_1} \Delta \underset{\sim}{\sigma} \cdot d[\Delta \underset{\sim}{\varepsilon}] + \underset{\sim}{\sigma}_o \cdot \Delta \underset{\sim}{\varepsilon}_1 \quad , \tag{4.46}$$

where

$$\Delta \underset{\sim}{\sigma}(t) \equiv \underset{\sim}{\sigma}(t) - \underset{\sim}{\sigma}_o \ , \quad \Delta \underset{\sim}{\varepsilon} = \int_{t_o}^{t} \dot{\underset{\sim}{\varepsilon}} dt \quad \text{for} \quad t_o \leq t \leq t_o + \Delta t \ . \tag{4.47}$$

Consider two different paths in $\Delta\underset{\sim}{\varepsilon}$-space, as depicted in Figure 3, namely, I_1: $\underset{\sim}{0} \rightarrow \Delta\underset{\sim}{\varepsilon}' \rightarrow \Delta\underset{\sim}{\varepsilon}$, and I_2: $\underset{\sim}{0} \rightarrow \Delta\underset{\sim}{\varepsilon}'' \rightarrow \Delta\underset{\sim}{\varepsilon}_1$. For each of these paths, let us evaluate the first integral on the right-hand side of Eq. (4.46):

$$\Delta W_1 = \int_{I_1} \Delta \underset{\sim}{\sigma} \cdot d[\Delta \underset{\sim}{\varepsilon}] \quad , \tag{4.48}$$

$$\Delta W_2 = \int_{I_2} \Delta \underset{\sim}{\sigma} \cdot d[\Delta \underset{\sim}{\varepsilon}] \quad . \tag{4.49}$$

Then

$$\Delta W_1 - \Delta W_2 = \left(\int_{I_1} - \int_{I_2} \right) \Delta \underset{\sim}{\sigma} \cdot d[\Delta \underset{\sim}{\varepsilon}] = \int_{I_1 - I_2} \Delta \underset{\sim}{\sigma} \cdot d[\Delta \underset{\sim}{\varepsilon}] \tag{4.50}$$

where the path $I_1 - I_2$ implies $\underset{\sim}{0} \rightarrow \Delta\underset{\sim}{\varepsilon}' \rightarrow \Delta\underset{\sim}{\varepsilon}_1 \rightarrow \Delta\underset{\sim}{\varepsilon}'' \rightarrow \underset{\sim}{0}$. Upon applying Stoke's Theorem in the non-dimensional strain increment space to transform the line integral to a surface integral (see Flanders, 1963), one obtains:

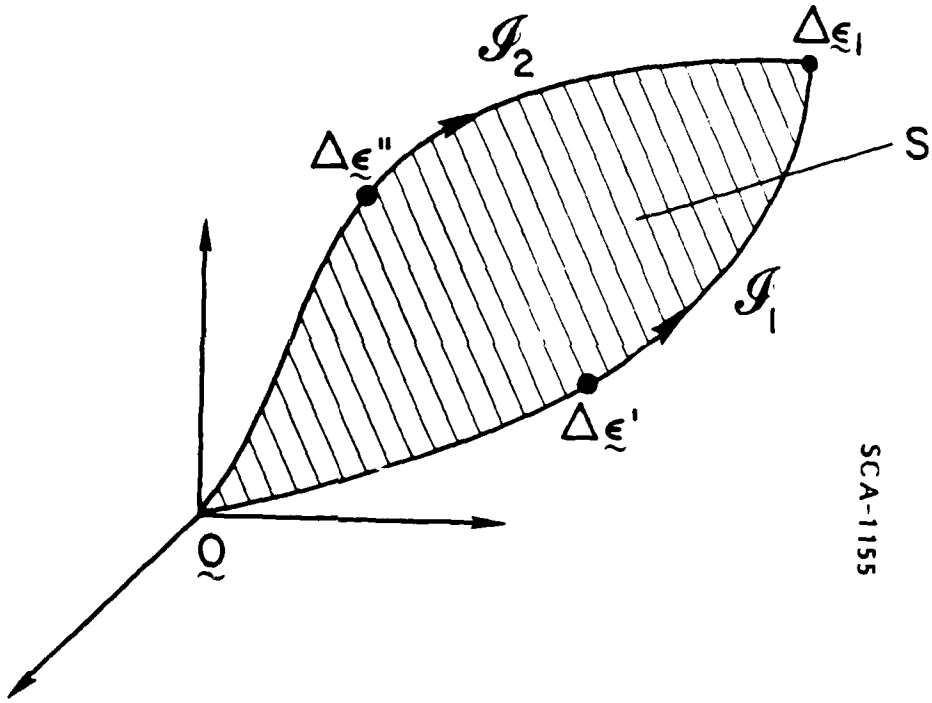

Figure 3. Different paths in $\Delta\underset{\sim}{\epsilon}$-space.

$$\Delta W_1 - \Delta W_2 = \int_{I_1 - I_2} \sum_{(i,j)} \Delta\sigma_{ij}\, d(\Delta\epsilon^{ij})$$

$$= \iint_S \sum_{(i,j)} \sum_{(k,\ell)} \left[\frac{\partial \Delta\sigma_{k\ell}}{\partial \Delta\epsilon^{ij}} - \frac{\partial \Delta\sigma_{ij}}{\partial \Delta\epsilon^{k\ell}}\right] d(\Delta\epsilon_{ij}) \wedge d(\Delta\epsilon_{k\ell}) \qquad (4.51)$$

when S is chosen so that the boundary of $S: \partial S = I_1 - I_2$ and $\wedge$ denotes the exterior product, i.e.,

$$d(\Delta\epsilon^{ij}) \wedge d(\Delta\epsilon^{k\ell}) = - d(\Delta\epsilon^{k\ell}) \wedge d(\Delta\epsilon^{ij}) \qquad (4.52)$$

and where $(i,j) < (k,\ell)$ implies that the pair (k,ℓ) follows the pair (i,j) in an established sequence of pairs.

In view of Eq. (4.40), the integrand of the integral in Eq. (4.51) is nonzero for non-associated flow rules and zero for associated flow rules. Therefore, for non-associated flow rules we may have

$$\Delta W_1 - \Delta W_2 \neq 0 \qquad (4.53)$$

while, for associated flow rules,

$$\Delta W_1 - \Delta W_2 = 0 \quad . \qquad (4.54)$$

This proves the path-dependency of the strain energy integral. It is important to note that the strain energy integral exists uniquely even for non-associated flow rules since there always exists one and only one $\dot{\underset{\sim}{\sigma}}$ for a given $\dot{\underset{\sim}{\epsilon}}$, as shown in Eq. (4.34).

In the incremental numerical approaches used in finite element and finite difference methods, it is tacitly assumed that $\Delta\underset{\sim}{\epsilon}_1$, is achieved in a proportional manner, that is, for any t such that $t_o \leq t \leq t_o + \Delta t$, we have

$$\Delta\underset{\sim}{\epsilon}(t) = \frac{t - t_o}{\Delta t}\, \Delta\underset{\sim}{\epsilon}_1 \quad . \qquad (4.55)$$

The implication of the path-dependency of ΔW for non-associated flow rules, therefore, is that in the numerical approaches one must use small time steps so that the proportional path approximation does not significantly deviate from the actual strain path. For associated flow rules, there is no concern since the strain path does not change the integral ΔW for a specified terminal strain increment, $\Delta \underset{\sim}{\varepsilon}$.

The above requirement for small time steps in the integration of the equations for non-associated plasticity is physical and independent of other considerations based upon numerical stability.

SECTION 5

UNIQUENESS

In this section, the question of uniqueness of initial-boundary value problems for elasto-plastic materials having non-associated flow rules is explored. Uniqueness is, of course, an issue that is completely unrelated to thermodynamics. Valanis (1985) showed that the usual approach for proving uniqueness for associated elasto-plastic materials is valid under the assumption of proportional paths even when the plastic flow is non-associated. However, when the requirement is not satisfied, the bifurcation and stability analyses require a special care for non-associated plasticity theories.

5.1 SUFFICIENT CONDITION.

Consider a three-dimensional continuum at time t having volume V and surface S. Let $\underset{\sim}{\sigma}(\underset{\sim}{x},t)$, $\underset{\sim}{\epsilon}(\underset{\sim}{x},t)$ and $\underset{\sim}{v}(\underset{\sim}{x},t)$ denote the stress field, strain field and velocity field, respectively, in V. In addition, let $\underset{\sim}{\sigma}(\underset{\sim}{x},0) = \underset{\sim}{0}$, $\underset{\sim}{\epsilon}(\underset{\sim}{x},0) = \underset{\sim}{0}$ and $\underset{\sim}{v}(\underset{\sim}{x},0) = \underset{\sim}{0}$ in V. Also, let S_T be that part of the surface over which the traction vector $\underset{\sim}{T}(t)$ is prescribed and S_D the remaining portion over which the displacement is prescribed in the time domain $0 < t < \infty$.

Restricting attention to small deformations, we proceed to explore the conditions under which the solution $\underset{\sim}{\sigma}(\underset{\sim}{x}t)$, $\underset{\sim}{\epsilon}(\underset{\sim}{x},t)$, and $\underset{\sim}{v}(\underset{\sim}{x},t)$ in V is unique, for $0 < t < \infty$, when the following equations apply:

In V:

$$\sigma_{ij,j} = \rho \dot{v}_i \quad , \tag{5.1}$$

$$\dot{\sigma}_{ij} = B_{ijk\ell} \dot{\epsilon}_{k\ell} \quad , \tag{5.2}$$

where (5.1) is the equation of motion in the absence of a body force and (5.2) is the basic constitutive equation for an elasto-plastic solid. For a non-associated flow rule, the inelastic tangent modulus, $B_{ijk\ell}$, is non-symmetric, i.e.,

$$B_{ijk\ell} \neq B_{k\ell ij} \quad , \tag{5.3}$$

a fact that will prove to be important in the developments which follow.

On S_T:

$$\sigma_{ij} n_j = T_i \ ; \tag{5.4}$$

On S_D:

$$v_i \text{ is prescribed} \ . \tag{5.5}$$

Consider now an incremental process from t_o to $t_1 = t_o + \Delta t$, and assume that a unique solution exists up to time t_o. In order to prove uniqueness by negation, we assume two solutions 1 and 2 of the incremental process and define the difference state between t_o and t_1:

$$\Delta v_i \equiv v_i^{(2)} - v_i^{(1)} \quad , \quad \Delta\sigma_{ji} \equiv \sigma_{ji}^{(2)} - \sigma_{ji}^{(1)} \tag{5.6}$$

for which the values at t_o and t_1 are prescribed:

$$\Delta v_i(t_o) = \Delta\sigma_{ji}(t_o) = 0 \quad ,$$

$$\Delta v_i(t_1) \equiv \Delta\bar{v}_i = v_i^{(2)}(t_1) - v_i^{(1)}(t_1) \quad , \tag{5.7}$$

$$\Delta\sigma_{ji}(t_1) \equiv \Delta\bar{\sigma}_{ji} = \sigma_{ji}^{(2)}(t_1) - \sigma_{ji}^{(1)}(t_1) \quad .$$

Since the two solution states satisfy (5.1), the difference state satisfies

$$\Delta\sigma_{ji,j} - \rho\Delta\dot{v}_i = 0 \quad . \tag{5.8}$$

Multiplying Eq. (5.8) by $(-\Delta v_i)$ and integrating over the domain V and time between t_o and t_1, one obtains

$$I \equiv \int_{t_o}^{t_1} \iiint_V \Delta v_i \left(\rho \Delta \dot{v}_i - \Delta \sigma_{ji,j} \right) dV \, dt$$

$$= \int_{t_o}^{t_1} \iiint_V \left[\left\{ \frac{\rho}{2} \left(\Delta \dot{v}_i \right)^2 \right\} - \left(\Delta \sigma_{ji} \Delta v_i \right)_{,j} + \Delta \sigma_{ji} \Delta v_{i,j} \right] dV dt = 0 \qquad (5.9)$$

Using the Gauss theorem and the boundary conditions for the difference state obtained from (5.4) and (5.5), Eq. (5.9) reduces to

$$I = \iiint_V \left\{ \frac{\rho}{2} \left(\Delta \bar{v}_i \right)^2 + \int_{t_o}^{t_1} \Delta \sigma_{ji} \Delta \dot{\bar{\epsilon}}_{ij} dt \right\} dV$$

$$= \iiint \left\{ \frac{\rho}{2} \left(\Delta \bar{v}_i \right)^2 + \int_o^{\Delta \bar{\epsilon}_{ij}} \Delta \sigma_{ji} d\left(\Delta \epsilon_{ij} \right) \right\} dV = 0 \quad . \qquad (5.10)$$

where

$$\Delta \bar{\epsilon}_{ij} = \left(\Delta \bar{v}_{i,j} + \Delta \bar{v}_{j,i} \right) / 2 \quad . \qquad (5.11)$$

The first integral of (5.10) defines the kinetic energy and is zero only when $\Delta \bar{v}_i = 0$.

For the associated flow problems with an incremental potential $\Delta \bar{W}$

$$\Delta \bar{\sigma}_{ij} = \frac{\partial \Delta \bar{W}}{\partial \Delta \bar{\epsilon}_{ij}} \; ; \quad \Delta \bar{W} = \int_o^{\Delta \bar{\epsilon}_{ij}} \Delta \sigma_{ij} d\left(\Delta \epsilon_{ij} \right) \qquad (5.12)$$

since the stress increment $\Delta \bar{\sigma}_{ij}$ is a unique function of $\Delta \bar{\epsilon}_{ij}$. Introducing $\Delta \bar{W}$ into (5.10), one obtains

$$I = \iiint_V \left\{ \frac{\rho}{2} \left(\Delta \bar{v}_i \right)^2 + \Delta \bar{W} \right\} dV = 0 \tag{5.13}$$

in which the kinetic energy of the difference state and $\Delta \bar{W}$ are both positive definite. Therefore, in order to satisfy (5.10), we must have:

$$\Delta \bar{v}_i = 0 \quad \text{and} \quad \Delta \bar{\epsilon}_{ij} = 0 \text{ everywhere in V.}$$

This is contrary to the assumption that $\Delta \bar{v}_i = 0$ and $\Delta \bar{\epsilon}_{ij} \neq 0$ in V. Therefore, uniqueness has been proven by negation.

The above procedure works well for associated flow plasticity in which a potential exists and Eq. (5.12) holds.

The approach requires special care, however, for non-associated flow rules. Since in this case $\Delta \sigma_{ij}$ is path-dependent, it is possible to have

$$\Delta \bar{V}_i = \Delta \bar{\epsilon}_{ij} = 0 \quad \text{and yet} \quad \Delta \bar{\sigma}_{ij} \neq 0 \tag{5.14}$$

for special path.

However, restricting to proportional paths in achieving $\Delta \bar{\epsilon}_{ij}$ and $\Delta \bar{\sigma}_{ij}$, the inequality

$$\Delta \bar{W} = \frac{1}{2} \Delta \bar{\sigma}_{ij} \Delta \bar{\epsilon}_{ij} > 0 \tag{5.15}$$

furnishes a sufficient condition for uniqueness but not a necessary condition even under quasi-static load environment (Valanis, 1985).

It should be noted that for sufficiently small Δt, the $\sigma_{ij} - \epsilon_{ij}$ path followed to achieve $\Delta \bar{\epsilon}_{ij}$ in an incremental process may be approximated by a piecewise incremental proportional path. Adopting this assumption, Mroz (1963;1966) developed a special class of non-associated flow models based upon the uniqueness requirement so that they possess the minimum hardening required to satisfy the conditions (5.15).

If $\Delta\bar{\epsilon}_{ij}$ is decomposed into its elastic and plastic components, the condition (5.15) can then be written in the form

$$\Delta\bar{\sigma}_{ij}\Delta\bar{\epsilon}_{ij} = \Delta\bar{\sigma}_{ij}\left(\Delta\bar{\epsilon}_{ij}\right)_e + \Delta\bar{\sigma}_{ij}\left(\Delta\bar{\epsilon}_{ij}\right)_p \geq 0 \quad . \tag{5.16}$$

The first term on the right-hand side of (5.16) denotes the incremental elastic strain energy and is positive definite. The second term, associated with the plastic deformation, becomes negative only in the case of non-associated flow laws for stress increments $\Delta\bar{\sigma}_{ij}$ whose direction θ falls within the range $\pi/2 < |\theta + \beta| < \pi$ (see Section 4.3).

Hill (1958) has shown that the requirements (5.15) is a sufficient, but not necessary, condition for stability. Also, the violation of this condition is known to cause some types of localized failure. The judgement of whether or not to accept such local failures as real or not should be based on experimental evidence.[****]

5.2 DISCUSSION.

In this section, the question of uniqueness for initial value problems involving non-associated elasto-plastic materials has been explored. We have shown that the usual mathematical approach for proving uniqueness of initial value problems involving elasto-plastic solids with associated flow laws holds even for solids with non-associated flow laws. In several cases, plasticity theories of non-associated flow laws have been constructed based upon the uniqueness requirement, such as for example those considered by Mroz (1963, 1966). Parenthetically, we note that the inability to prove uniqueness should not be taken to imply that uniqueness does not exist; it simply means that we are unable to answer the question of uniqueness. To address uniqueness of non-associated flow problems, a new mathematical approach is needed, which is beyond the scope of this study. At this moment, it is interesting to note the inadequacy of associated flow plasticity in predicting localized failure, such as necking, in metals. For example, the numerical study of necking in plane strain tensile test by McMeeking and Rice (1975) which employed a J_2 associated flow theory found continued growth of the diffuse neck, with no tendency toward intense localized shearing. Successful simulations by Tvergaard, Needleman and Lo (1981), and Iwakuma and Nemat-Nasser (1982) utilized the flow theory which includes non-

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**** For further discussion of destabilizing effects of non-associated flow laws, see Maier and Hueckel (1979).

coaxiality. For pressure-sensitive geomaterials, Rudnicki and Rice (1975) utilized the nonuniqueness of non-associated flow theories to predict the localization of deformation as a constitutive instability (bifurcation). The above examples indicate that the non-associated flow rules which exhibit non-normality and non-coaxiaility of stress tensor and plastic strain rate tensor may be of necessity to predict the inception of unstable deformation at realistic stress level.

SECTION 6

APPLICATION TO SOME NON-ASSOCIATED MODELS HAVING J_2-FLOW POTENTIALS

In Section 4.1, the restrictions imposed on non-associated flow models by thermodynamics were explored. It was shown that the second law of thermodynamics is satisfied whenever the inequality given earlier in Eq. (4.6), i.e.,

$$\frac{\partial f}{\partial \underset{\sim}{\sigma}} \cdot \underset{\approx}{L} \cdot \frac{\partial \phi}{\partial \underset{\sim}{\sigma}} \geq 0 \tag{6.1}$$

is satisfied. The issue of uniqueness, discussed in Section 5, was much less conclusive.

In this section, we consider two non-associated flow models and investigate the restrictions, if any, imposed upon them by the second law requirement (6.1). The specific models were selected for examination because they represent two important classes of models presently being used by the defense community in ground motion simulation studies. Both models are elastic-perfectly plastic, and thus do not exhibit strain hardening. Also, they both have pressure-dependent yield surfaces with J_2-plastic potentials.

6.1 GENERALIZED DRUCKER-PRAGER MODEL.

This model has a pressure dependent yield surface of the form:

$$f(\underset{\approx}{s}, p) = J_2 - F(p) \quad , \tag{6.2}$$

and a von Mises-type plastic potential, i.e.

$$\phi(\underset{\approx}{s}) = J_2 \tag{6.3}$$

where $\underset{\approx}{s}$ denotes the deviatoric stress tensor, p is the pressure, and

$$J_2 = \frac{1}{2} s_{ij} s_{ij} \quad . \tag{6.4}$$

The function F is positive and increases monotonically with p.
The deviatoric response follows an incremental form of Hooke's Law

$$\dot{s}_{ij} = 2\mu\left(\dot{e}_{ij} - \dot{e}^{p}_{ij}\right) \tag{6.5}$$

where $\underset{\sim}{e}$ and $\underset{\sim}{e}^{p}$ denote, respectively, the deviatoric strain tensor and the plastic deviatoric strain tensor, while μ denotes the shear modulus. The hydrostatic behavior follows a hypoelastic expression of the form:

$$\dot{p} = K(\epsilon)\dot{\epsilon} \tag{6.6}$$

where p is the pressure, ϵ represents the volumetric strain, and K denotes the tangent bulk modulus.

From Eqs. (6.5) and (6.6), L in Eq. (4.2) may be expressed as:

$$L_{ijk\ell} = \left\{K(\epsilon) - \frac{2}{3}\mu\right\}\delta_{ij}\delta_{k\ell} + 2\mu\delta_{ik}\delta_{j\ell} \quad . \tag{6.7}$$

Upon substituting Eqs. (6.2), (6.3) and (6.7) into the left-hand side of Eq. (6.1), one obtains the result:

$$\frac{\partial f}{\partial \sigma_{ij}} L_{ijk\ell} \frac{\partial \phi}{\partial \sigma_{k\ell}} = 4\mu J_2 \quad . \tag{6.8}$$

Thus, since, $J_2 \geq 0$, the model satisfies the condition (6.1) without any restrictions.

6.2 A THIRD INVARIANT GENERALIZED DRUCKER-PRAGER MODEL.

This model is similar to that described in the preceding section except that the yield surface is of the form:

$$f\left(\underset{\sim}{s}, p\right) = J_2 - \frac{\alpha}{F(p)} J_3 - F^2(p) = 0 \tag{6.9}$$

where F(p) is a positive, monotonically increasing function of p, $0 < \alpha < 1$, and

$$J_3 = \frac{1}{3} s_{ij} s_{jk} s_{ki} \quad . \tag{6.10}$$

The flow potential is given by Eq. (6.3), and Eqs. (6.5) and (6.6) also apply, with L defined by Eq. (6.7). It is then straightforward to show that the left-hand side of Eq. (6.1) takes the form:

$$\frac{\partial f}{\partial \sigma_{ij}} L_{ijk\ell} \frac{\partial \phi}{\partial \sigma_{k\ell}} = 2\mu \left(2J_2 - 3 \frac{\alpha}{F(p)} J_3 \right) \equiv I \quad . \tag{6.11}$$

To prove the Eq. (6.11) is positive for $0 < \alpha < 1$, let us introduce the variables r,θ in the π-plane as follows:

$$r^2 = 2J_2 = s_1^2 + s_2^2 + s_3^2 \quad , \tag{6.12}$$

$$\cos 3\theta = \frac{3\sqrt{3}}{2} \frac{J_3}{(J_2)^{3/2}} \quad . \tag{6.13}$$

Equation (6.9) can then be rewritten as:

$$f = \frac{r^2}{2} - \frac{\alpha}{F(p)} \frac{r^3}{3\sqrt{6}} \cos 3\theta - F^2(p) = 0 \quad . \tag{6.14}$$

On $f = 0$ in the π-plane, r is strictly a function of θ and

$$\frac{\partial r}{\partial \theta} = 0 \quad \text{at} \quad \sin 3\theta = 0 \quad . \tag{6.15}$$

Also, for $\cos 3\theta = 0$, it follows from Eq. (6.14) that

$$\frac{r}{F(p)} = \sqrt{2} \quad , \tag{6.16}$$

as depicted in Figure 4.

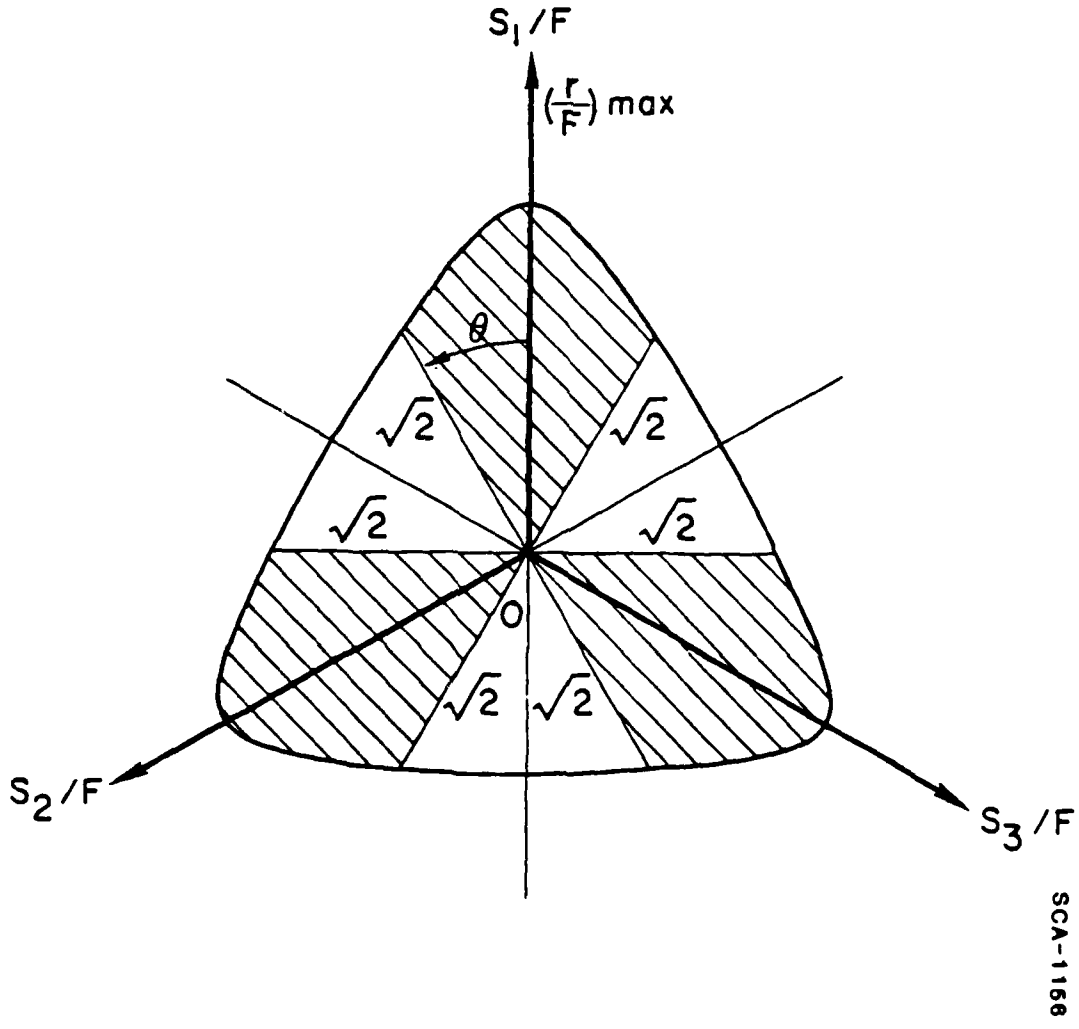

Figure 4. Trace of third invariant yield surface in the Π-plane, showing the manner in which θ is defined.

Equation (6.11) may be rewritten, using Eqs. (6.12) and (6.13) as:

$$I = 2\mu r^2\left\{1 - \frac{\alpha}{\sqrt{6}}\frac{r}{F(p)}\cos 3\theta\right\} \quad .$$

For $\cos 3\theta \leq 0$, it is trivial to show that $I > 0$. To prove that $I > 0$ for $\cos 3\theta > 0$, which includes $\theta = 0, \pm 2\pi/3$ at which r takes maximum values (see Eq. (6.15)), first note that

$$\left(\frac{r}{F(p)}\right)_{max} < \sqrt{6} \quad \text{for } 0 \leq \alpha < 1 \tag{6.18}$$

which, when used in Eq. (6.18), leads to the result:

$$I \geq 2\mu r^2\left\{1 - \frac{\alpha}{\sqrt{6}}\left[\frac{r}{F(p)}\right]_{max}\right\} \geq 2\mu r^2(1-\alpha) > 0 \quad . \tag{6.19}$$

Thus, the model satisfies the basic thermodynamic requirement, Eq. (6.1), without any restrictions.

SECTION 7

CONCLUSIONS

In this study, we have explored the possible restrictions that thermodynamics and uniqueness impose on problems involving non-associated elasto-plastic solids. From this investigation, the following major conclusions have emerged:

1. In the case of elasto-plastic constitutive models formulated by defining a plastic potential ϕ, a yield surface f and a non-associated flow rule, i.e.,

$$\dot{\underset{\sim}{\varepsilon}}_p = \lambda \frac{\partial \phi}{\partial \underset{\sim}{\sigma}} \quad ,$$

the only restriction imposed by thermodynamics is that the plastic *multiplier* λ *must* be positive, i.e., $\lambda > 0$.

2. If an inelastic constitutive model is constructed by defining the Helmholtz free energy function ψ and the evolutionary equations for the internal variables, $\underset{\sim}{\xi}$, thermodynamic considerations then require that the following condition be satisfied:

$$-\frac{\partial \psi}{\partial \underset{\sim}{\xi}} \cdot \dot{\underset{\sim}{\xi}} \geq 0 \quad .$$

This condition usually leads to restrictions on the form of the evolutionary equations for the internal variables $\underset{\sim}{\xi}$.

3. The postulates of Drucker and Il'iushin are shown to be unrelated to the second law of thermodynamics. It was found that these postulates can be violated for processes in which the second law is satisfied.

4. Non-associated flow laws satisfy the continuity condition of Prager (1949).

5. Elasto-plastic solids having non-associated flow laws exhibit a seemingly peculiar response feature in which the directional plastic modulus can exceed the corresponding elastic modulus for a certain range of stress increment vectors in stress space. For this response, nonunique solutions may exist, which suggest the inception of localized failure due to bifurcation.

6. One of the major differences between associated and non-associated flow laws is that associated flow laws are not path-dependent while non-associated flow laws are path-dependent.

7. Because of the path-dependent nature of non-associated flow laws, the approach for proving uniqueness for initial value problems involving associated flow laws remains valid for exploring uniqueness under the proportional-path assumption even if the flow law is non-associated.

8. Uniqueness has been proven only for a special class of non-associated flow laws in which there is sufficient hardening to eliminate destabilizing effects. For non-associated model with no strain hardening, uniqueness cannot be proven by following the usual approach without performing bifurcation and stability analyses. A new mathematical approach is needed for exploring uniqueness of non-associated flow problems.

9. Two generic types of non-associated elasto-plastic models that are commonly used by the defense community to simulate explosive ground motion were considered in detail. It was shown that both types of models satisfy the second law as is, without the need for further restriction. On the other hand, since both models are perfectly plastic (no hardening), there is a potential for instability and possibly a lack of uniqueness. Also, care must be taken in numerically dealing with non-associated flow rules since they lead to non-selfadjoint operators, which are known to require special treatment.

On the basis of the above remarks, it follows that the greatest concerns in using non-associated flow models without hardening arise from (1) the uncertainty of the question of uniqueness, (b) possible instabilities, (c) numerical difficulties in dealing with non-selfadjoint operators and (d) the seemingly peculiar feature of response in which the directional plastic modulus can exceed the corresponding elastic modulus when instability is anticipated. Further research is clearly needed to resolve these issues before non-associated flow laws can be used with reasonable confidence.

SECTION 8

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